数理解析研究所講究録
第 1886 巻 2014 年 181-188

Evaluation of the Scale Risk

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1 Introduction

We study the evaluation problem of the scle risk. The method we have adopted is the risk sensitive value measure (RSVM) method, which have been introduced in [8].

This method is developed originally for the project evaluation. Even so this method can be applied to many evaluation problems in finance. For example we can apply this method to the scale risk evaluation problems.

In this paper we overview the idea of the scale risk evaluation problem. For the details, see [8] and etc.

2 Risk-Sensitive Value Measure(RSVM)

We give the definition of the Risk-Sensitive Value Measure(RSVM) and summarize the properties of this measure.

2.1 Definition of the Risk-Sensitive Value Measure

Definition 1 (Risk sensitive value measure(RSVM)) *Let* $\mathbf{X}$ *be a linear space of random variables, then the risk sensitive value measure(RSVM) on* $\mathbf{X}$ *is the following functional defined on* $\mathbf{X}$

$$U^{(\alpha)}(X)=-\frac{1}{\alpha}\log E[e^{-\alpha X}], \quad (\alpha>0), \tag{2.1}$$

where α *is the risk avertion parameter.*

Remark 1 *In the above definition,* X *is supposed to be the random present value of a cash flow or a return of some asset.*

2.2 Properties of the Risk-Sensitive Value Measure

We first remark the following facts.

Proposition 1 (i) *The following approximation formula holds true:*

$$U^{(\alpha)}(X)=E[X]-\frac{1}{2}\alpha V[X]+\cdots . \tag{2.2}$$

(ii) *If* X *is Gaussian, then it holds that*

$$U^{(\alpha)}(X)=E[X]-\frac{1}{2}\alpha V[X]. \tag{2.3}$$

2.2.1 Concave Monetary Value Measure

Definition 2 (concave monetary value measure) *A function $v(\cdot)$ defined on a linear space* $\mathbf{X}$ *of random variables is called* a concave monetary value measure (or concave monetary utility function) *on* $\mathbf{X}$ *if it satisfies the following conditions:*

(i) (Normalization) : $v(0) = 0$,

(ii) (Monetary property) : $v(X + m) = v(X) + m$, where m is non-random,
(Remark: (i) + (ii) $\rightarrow v(m) = m$),

(iii) (Monotonicity) : If $X \geq Y$, then $v(X) \geq v(Y)$,

(iv) (Concavity) : $v(\lambda X + (1-\lambda)Y) \geq \lambda v(X) + (1-\lambda)v(Y)$ for $0 \leq \lambda \leq 1$,

(v) (Law invariance) : $v(X) = v(Y)$ whenever law(X) = law(Y),

Remark 2 *We don't require the following positive homogeneity property:*
(vi) (Positive Homogeneity): $\forall \lambda \in R^+, \quad v(\lambda X) = \lambda v(X)$.

We next notice an important property of a concave monetary value measure.

Proposition 2 (global concavity) *A concave monetary value measure $v(\cdot)$ satisfies the following global concavity condition.*
(iv) ' (global concavity) :
$v(\lambda X + (1-\lambda)Y) \leq \lambda v(X) + (1-\lambda)v(Y)$ for $\lambda \leq 0$ or $\lambda \geq 1$

Proposition 3 *Let $v(\cdot)$ be a concave monetary value measure. Then, for a fixed pair (X, Y), $\psi_{X,Y}(\lambda) = v(\lambda X + (1-\lambda)Y)$ is a concave function of λ.*

Setting $Y = 0$ in this proposition, we obtain the following result:

Corollary 1 *Let $v(\cdot)$ be a concave monetary value measure. Then $\psi_X(\lambda) = v(\lambda X)$ is a concave function of λ and $\psi_X(0) = 0$.*

From this corollary we obtain the following concept of "Optimal Scale."
[Optimal Scale]
Let $v(\cdot)$ be a concave monetary value measure, and assume that $v(X_0) > 0$ for some fixed random variable X_0. If $v(\lambda X_0), \lambda > 0$, is an upper bounded function of λ, then we can find the maximum point $\bar{\lambda}$. This value $\bar{\lambda}$ is the optimal scale of X_0.

2.2.2 Utility Indifference Value

For a utility indifference value we obtain the following result:

Proposition 4 *Let $u(x)$ be a utility function defined on $(-\infty, \infty)$ and satisfy the usual properties of a utility function. Then the indifference value $v(X)$ determined by the following equation*

$$E[u(-v(X) + X)] = u(0) = 0 \tag{2.4}$$

is a concave monetary value measure.

Remark 3 *An indifference value does not satisfy the following positive homogeneity condition in general.*
(Positive Homogeneity): $\forall\lambda \in R^+, \quad v(\lambda X) = \lambda v(X).$

Proposition 5 *$U^{(\alpha)}(X)$ is the indifference value of the exponential utility function:*

$$u_\alpha(x) = \frac{1}{\alpha}\left(1 - e^{-\alpha x}\right), \quad -\infty < x < \infty \qquad (\alpha > 0). \tag{2.5}$$

Corollary 2 *$U^{(\alpha)}(X)$ is a concave monetary value measure.*

Corollary 3 *$U^{(\alpha)}(\lambda X)$ is a concave function of λ.*

2.2.3 Optimal Scale

From the fact that $U^{(\alpha)}(\lambda X)$ is a concave function of λ, we can discuss the optimal scale of the investment, and we obtain the following result:

Proposition 6 *Assume that the moment generation function of X converges and that the following conditions satisfied,*

$$E[X] > 0, \quad P(X < 0) > 0. \tag{2.6}$$

Then it holds that
(i) *When λ (> 0) is small, $U^{(\alpha)}(\lambda X) > 0$, and*

$$\lim_{\lambda\to\infty} U^{(\alpha)}(\lambda X) = -\infty. \tag{2.7}$$

(ii) *The optimal scale λ_{opt} is*

$$\lambda_{opt} = \frac{C_X}{\alpha}, \quad \alpha > 0, \tag{2.8}$$

where C_X is a solution of $E\left[Xe^{-C_X X}\right] = 0$.

2.2.4 Independence-Additivity Property

Definition 3 (Independence-Additivity) *If a value measure $v(\cdot)$ satisfies*
e) (independence-additivity): $v(X + Y) = v(X) + v(Y)$ *if X and Y are independent, then $v(\cdot)$ is said to have the independence-additivity property.*

We can suppose that this property is desirable for the project evaluation functional, and the following proposition is easily proved.

Proposition 7 *An indifference value determined from an exponential utility function has the independence-additivity property.*

The converse of this proposition is known.

Proposition 8 *Let $v(x)$ be an indifference value determined by a utility function $u(x)$ which is of $C^{(2)}$-class, increasing, concave, and normalized such as $u(0) = 0$, $u'(0) = 1$, and $u''(0) = \alpha$. Then, if $v(x)$ has the independence-additivity property, $u(x)$ is of the following form*

$$u(x) = u_\alpha(x) = \frac{1}{\alpha}\left(1 - e^{-\alpha x}\right). \tag{2.9}$$

2.3 Good Points of Risk Sensitive Value Measure

(1) The RSVM is a concave monetary value measure.
(2) The RSVM is the utility indifference value of the exponential utility function, and it has a risk aversion parameter α.
(3) The optimal scale of a project can be discussed.
(4) The RSVM has the independence-additivity property, and the RSVM is the almost only one which has this property in the set of all utility indifference values.
(5) The dynamic RSVM has the time-consistency property, and the RSVM is the almost only one which has this property in the set of all utility indifference values.

3 Evaluation of the Scale Risk

3.1 What is the Scale Risk

Let X be a return for an investment of I. We suppose that the return for the investment λI is λX. Assume that $E[X] > 0$ and $P(X < 0) > 0$. If $\lambda(> 0)$ is small then the investment λI may be positively valued. But if λ is very large, then a very big loss may happen and so the investment λI may be negatively valued. This is the "scale risk."

3.2 Numerical Example

Let X, Y, Z be random variables whose distributions are

$$P(X = -10) = 0.02, \quad P(X = 4) = 0.5, \quad P(X = 8) = 0.48 \tag{3.1}$$
$$E[X] = 5.64, \quad V[X] = 8.9104, \tag{3.2}$$
$$P(Y = -2) = 0.15, \quad P(Y = 4) = 0.7, \quad P(Y = 10) = 0.15 \tag{3.3}$$
$$E[Y] = 4.00, \quad V[Y] = 10.8000, \tag{3.4}$$
$$P(Z = -1) = 0.3, \quad P(Z = 4) = 0.6, \quad P(Z = 16) = 0.1 \tag{3.5}$$
$$E[Z] = 3.70, \quad V[Z] = 21.8100. \tag{3.6}$$

From the scale risk point of view, X has a big scale risk, Z has a less scale risk and Y is between X and Z. Remak here also that

$$E[X] > E[Y] > E[Z] \tag{3.7}$$

$$V[X] < V[Y] < V[Z] \tag{3.8}$$

We calculate the values of λX, λY and λZ. In the following table,

$$MV_X(\lambda) = E[\lambda X] - \tfrac{1}{2}\alpha V[\lambda X], \quad RSVM_X(\lambda) = U^{(\alpha)}(\lambda X), \tag{3.9}$$
$$MV_Y(\lambda) = E[\lambda Y] - \tfrac{1}{2}\alpha V[\lambda Y], \quad RSVM_Y(\lambda) = U^{(\alpha)}(\lambda Y), \tag{3.10}$$
$$MV_Z(\lambda) = E[\lambda Z] - \tfrac{1}{2}\alpha V[\lambda Z], \quad RSVM_Z(\lambda) = U^{(\alpha)}(\lambda Z), \tag{3.11}$$

where MV_X is the mean variance value of X.

$\alpha = 0.05$

λ	MV_X	$RSVM_X$	MV_Y	$RSVM_Y$	MV_Z	$RSVM_Z$
1	5.417240	5.381304	3.730000	3.729802	3.154750	3.213878
2	10.388960	10.043808	6.920000	6.917064	5.219000	5.649037
3	14.915160	13.521364	9.570000	9.556959	6.192750	7.511068
4	18.995840	15.127878	11.680000	11.646280	6.076000	8.922791
5	22.631000	14.163164	13.250000	13.188966	4.868750	9.959279
6	25.820640	10.355244	14.280000	14.200910	2.571000	10.671750
7	28.564760	4.096341	14.770000	14.712311	-0.817250	11.100837
8	30.863360	-3.853309	14.720000	14.766822	-5.296000	11.282718
9	32.716440	-12.796062	14.130000	14.417932	-10.865250	11.251235
10	34.124000	-22.268194	13.000000	13.723921	-17.525000	11.038123
11	35.086040	-32.008482	11.330000	12.742895	-25.275250	10.672577
12	35.602560	-41.881393	9.120000	11.528915	-34.116000	10.180772
13	35.673560	-51.819265	6.370000	10.129651	-44.047250	9.585588
14	35.299040	-61.788863	3.080000	8.585410	-55.069000	8.906588
15	34.479000	-71.773961	-0.750000	6.929195	-67.181250	8.160181
16	33.213440	-81.766643	-5.120000	5.187368	-80.384000	7.359922
17	31.502360	-91.763044	-10.030000	3.380592	-94.677250	6.516870
18	29.345760	-101.761270	-15.480000	1.524834	-110.061000	5.639959
19	26.743640	-111.760395	-21.470000	-0.367698	-126.535250	4.736348
20	23.696000	-121.759963	-28.000000	-2.287744	-144.100000	3.811738

From the above table we can see that the RSVM is a desirable value measure which contains the evaluation of scale risk.

4 Hedging of the Scale Risk

A numerical example

Let X and W be given as follows,

$$P(\{\omega_1\}) = 0.02,\ P(\{\omega_2\}) = 0.5,\ P(\{\omega_3\}) = 0.48, \tag{4.1}$$

$$X(\omega_1) = -10, X(\omega_2) = 4, X(\omega_3) = 8; \quad E[X] = 5.64, V[X] = 8.9104, \tag{4.2}$$

$$W(\omega_1) = 10, W(\omega_2) = -1, W(\omega_3) = -1; \quad E[W] = -0.7800, V[W] = 2.3716. \tag{4.3}$$

(The distribution of X is same as before.)

Then we obtain

$$U^{(0.05)}(X) = 5.381304 > 0, \quad U^{(0.05)}(10X) = -22.268194 < 0 \tag{4.4}$$

$$U^{(0.05)}(W) = -0.8301 < 0, \quad U^{(0.05)}(10W) = -9.5976 < 0. \tag{4.5}$$

So, X may be carried out but $10X$, W and $10W$ are not carried out.

On the other hand, we obtain the following results,

$$U^{(0.05)}(X+W) = 4.7498 > 0, \ U^{(0.05)}(10X+10W) = 38.4748 > 0. \tag{4.6}$$

Therefore, both $X + W$ and $10X + 10W$ may be carried out. This means that W or $10W$ are valueless, but we can hedge the scale risk of $10X$ by the use of $10W$.

5 Inner Rate of Risk Avertion(IRRA)

5.1 Definition of the Inner Rate of Risk Avertion (IRRA)

Definition 4 *Let X be an asset. Then a solution α of the following equation*

$$U^{(\alpha)}(X) = 0 \tag{5.1}$$

is called the inner rate of risk avertion (IRRA) of X, and denoted by $\alpha_0(X)$.

Remark 4 *The larger $\alpha_0(X)$ is, the smaller the risk of X is. So the IRRA can be a rating index of assets.*

5.2 Existence of the IRRA

For the existence of IRRA, we obtain the following result:

Proposition 9 *Assume that the moment generation function of a random variable X converges, and the following conditions satisfied,*

$$E[X] > 0 \quad \text{and} \quad P(X < 0) > 0. \tag{5.2}$$

Then the IRRA $\alpha_0(X)$ of X exists and is unique.

6 Concluding Remarks

The books and articles relating to this paper are listed in the References. ([1, 2, 3, 5, 6, 7, 8, 9, 10, 14, 15])

[Problems to which the Risk-Sensitive Value Measure Method can be Applied]
(1) Project evaluation.
(2) Evaluation of financial (or real) assets.
(3) Evaluation of big projects (energy or resources exploitation).
(4) Evaluation of research projects.
(5) Evaluation of the intellectual property.
(6) Evaluation of the credit risk.
(7) Evaluation of a portfolio.
(8) Evaluation of a company.

The papers, [4], [11], [12] , [13] are relating to those applications.

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